

# INTEGRALITY OF VOLUMES OF REPRESENTATIONS

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**ABSTRACT.** Let  $M$  be an oriented complete hyperbolic  $n$ -manifold of finite volume. Using the definition of volume of a representation previously given by the authors in [BBI13] we show that the volume of a representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^n)$ , properly normalized, takes integer values if  $n = 2m \geq 4$ . Moreover we prove that the volume is continuous in all dimension and hence, if  $\dim M = 2m \geq 4$ , it is constant on connected components of the representation variety.

If  $M$  is not compact and  $\dim M = 3$ , the volume is not locally constant and we give explicit examples of representations with volume as arbitrary as the volume of hyperbolic manifolds obtained from  $M$  via Dehn fillings.

## 1. INTRODUCTION

Let  $M = \Gamma \backslash \mathbb{H}^n$  be an oriented complete hyperbolic manifold of finite volume where  $\Gamma < \text{Isom}^+(\mathbb{H}^n)$  is a torsion-free lattice in the group of orientation preserving isometries of real hyperbolic  $n$ -space  $\mathbb{H}^n$ .

If  $M$  is compact, given a representation  $\rho : \Gamma \rightarrow \text{Isom}^+(\mathbb{H}^n)$  there is a classical notion of volume  $\text{Vol}(\rho)$  of  $\rho$ , admitting various definitions. We recall here the cohomological one. Corresponding to the volume form on  $\mathbb{H}^n$ , there is a continuous class  $\omega_n \in H_c^n(\text{Isom}^+(\mathbb{H}^n), \mathbb{R})$ ; the volume of  $\rho$  is then

$$\text{Vol}(\rho) := \langle \rho^*(\omega_n), [M] \rangle,$$

where  $[M]$  is the generator of  $H_n(M, \mathbb{R}) \cong H_n(\Gamma, \mathbb{R})$  given by the fundamental class and  $\rho^*(\omega_n) \in H^n(\Gamma, \mathbb{R})$  is the pullback of  $\omega_n$ .

This definition has been extended to the non-compact case by various authors [Dun99, Fra04, BIW10, KK12a, BBI13], and the equivalence of these definitions has been recently established in [KK13]. We will use the cohomological definition introduced in [BBI13], which parallels the definition for surface groups given in [BIW10], and we briefly recall it here (for more details see § 4). To account for the fact that if  $M$  is not compact the cohomology group  $H^n(\pi_1(M), \mathbb{R})$  vanishes,

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we resort to relative bounded cohomology. In fact, the volume form defines also a continuous bounded class  $\omega_n^b \in H_{cb}^n(\text{Isom}^+(\mathbb{H}^n), \mathbb{R})$ . We consider moreover  $\rho$  as a representation of the fundamental group  $\pi_1(N)$  of a compact core  $N$  of  $M$ , and obtain a bounded singular class

$$\rho^*(\omega_n^b) \in H_b^n(\pi_1(N), \mathbb{R}) \cong H_b^n(N, \mathbb{R}).$$

Since all boundary components of  $N$  have amenable fundamental group, there is an isometric isomorphism [BBF<sup>+</sup>12, KK12b]

$$j : H_b^n(N, \partial N, \mathbb{R}) \longrightarrow H_b^n(N, \mathbb{R}),$$

where the left hand side is the bounded cohomology of  $N$  relative to its boundary; this leads us to define

$$\text{Vol}(\rho) := \langle j^{-1}\rho^*(\omega_n^b), [N, \partial N] \rangle,$$

where  $[N, \partial N]$  is the relative fundamental class.

With this definition it is easy to deduce that

$$(1.1) \quad |\text{Vol}(\rho)| \leq \text{vol}(M).$$

One of the fundamental results concerning the volume of a representation is the volume rigidity theorem, according to which equality in (1.1) holds if and only if:

- (1)  $n = 2$  and  $\rho$  is the holonomy representation of a (possibly infinite volume) complete hyperbolization of the smooth surface underlying  $M$ , [Gol80, BI07, KM08], or
- (2)  $n \geq 3$  and  $\rho$  is conjugate to  $\text{Id}_\Gamma$ , [Dun99, FK06, BCG07, BBI13].

In this paper we will be mainly interested in the nature of the values of  $\text{Vol}$ .

If  $M$  is compact, one knows at least since [Rez96] that  $\text{Vol}$  is constant on the connected components of  $\text{Hom}(\pi_1(M), \text{Isom}^+(\mathbb{H}^n))$  and hence takes only finitely many values. In odd dimension the nature of these values is in general mysterious, while in even dimension  $n = 2m$ , the Chern–Gauss–Bonnet theorem [Spi79, Chapter 13, Theorem 26] implies that the volume  $\omega_n$  comes from a characteristic class and

$$\frac{2\text{Vol}(\rho)}{\text{vol}(S^{2m})} \in \mathbb{Z}.$$

If  $M$  is non-compact, the situation parallels the one above, at least in high dimension. In fact, using an approach via Schläfli’s formula as in [BCG07], Kim and Kim proved that if  $M$  is a finite volume hyperbolic manifold of dimension  $\geq 4$ , the volume is constant on the connected components of  $\text{Hom}(\pi_1(M), \text{Isom}^+(\mathbb{H}^n))$  [KK13]. Like in the compact case, in odd dimension the nature of these values is mysterious.

Our main result is the integrality of  $\text{Vol}(\rho)$  in dimension  $n = 2m \geq 4$ ; this generalizes the Harder–Gauss–Bonnet theorem [Har71] according to which

$$\frac{2(-1)^m}{\text{vol}(S^{2m})} \text{vol}(M) = \chi(M).$$

In the statement of the theorem we will have to consider different cases according to the structure of the cusps. Recall that a cusp cross section of a hyperbolic  $n$ -manifold is a compact flat  $(n-1)$ -manifold; the Bieberbach number  $B_{n-1}$  is then the smallest natural number such that any flat compact  $(n-1)$ -manifold has a  $B_{n-1}$ -covering that is a torus. It is known [Nim98] that every flat compact 3-manifold appears, up to diffeomorphisms, as a cusp cross section of a finite volume hyperbolic 4-manifold, while in [LR02] it was shown that every flat compact  $(n-1)$ -manifold is diffeomorphic to a cusp cross section of some finite volume hyperbolic  $n$ -orbifold (see also the discussion in [KM13]).

In this spirit we extend the notion of volume of a representation  $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$  to an arbitrary lattice  $\Gamma < \text{Isom}(\mathbb{H}^n)$  by setting

$$\text{Vol}(\rho) = \frac{\text{Vol}(\rho|_{\Gamma'})}{[\Gamma : \Gamma']},$$

where  $\Gamma' < \Gamma$  is a torsion free finite index subgroup.

**THEOREM 1.1.** *Let  $n = 2m \geq 4$  be an even integer. Let  $\Gamma < \text{Isom}^+(\mathbb{H}^{2m})$  be a non-cocompact lattice and let  $\rho : \Gamma \rightarrow \text{Isom}^+(\mathbb{H}^{2m})$  be any representation.*

- (1) *If  $\Gamma$  is torsion free and the manifold  $M = \Gamma \backslash \mathbb{H}^{2m}$  has only toric cusps, then*

$$\frac{2}{\text{vol}(S^{2m})} \cdot \text{Vol}(\rho) \in \mathbb{Z}.$$

- (2) *If  $\Gamma$  is torsion free then*

$$\frac{2}{\text{vol}(S^{2m})} \cdot \text{Vol}(\rho) \in \frac{1}{B_{2m-1}} \cdot \mathbb{Z},$$

*where  $B_{2m-1}$  is the Bieberbach number in dimension  $2m-1$ .*

- (3) *There exists an integer  $B' = B'(\Gamma) \geq 1$  such that*

$$\frac{2}{\text{vol}(S^{2m})} \cdot \text{Vol}(\rho) \in \frac{1}{B'} \cdot \mathbb{Z}.$$

It follows from (1.1) and Theorem 1.1 that, in fact,  $\frac{2}{\text{vol}(S^{2m})} \cdot \text{Vol}(\rho)$  takes only a finite number of values.

In contrast with Theorem 1.1, we prove also the following result, which will be used several times through the paper:

**THEOREM 1.2.** *Let  $\Gamma < \text{Isom}(\mathbb{H}^n)$  be any lattice. The function*

$$\begin{aligned} \text{Hom}(\Gamma, \text{Isom}(\mathbb{H}^n)) &\longrightarrow \mathbb{R} \\ \rho &\longmapsto \text{Vol}(\rho) \end{aligned}$$

is continuous.

As a consequence we deduce indeed immediately that the values of  $\text{Vol}$  are constant on the connected components of the representation variety  $\text{Hom}(\Gamma, \text{Isom}(\mathbb{H}^{2m}))$  for  $2m \geq 4$ .

The only question remaining is what are the possible values of  $\text{Vol}(\rho)$  if  $M$  is 3-dimensional and non-compact. In fact, if  $n = 2$ , we know that the function  $\rho \mapsto \frac{1}{2\pi} \text{Vol}(\rho)$  takes all the values in the interval  $[\chi(M), -\chi(M)]$ , [BIW10].

If  $n = 3$ , the character variety of  $\Gamma < \text{Isom}^+(\mathbb{H}^3)$  is smooth near  $\text{Id}_\Gamma$ , and its complex dimension near  $\text{Id}_\Gamma$  equals the number  $h$  of cusps of  $M$  [Thu78]. As a result of the volume rigidity theorem and the continuity of  $\text{Vol}$ , the image of  $\text{Vol}$  contains at least an interval  $(\text{vol}(M) - \epsilon, \text{vol}(M)]$ . Special points in the character variety of  $\Gamma$  come from Dehn fillings of  $M$ . Let  $M_\tau$  denote the compact manifold obtained from  $M$  by Dehn surgery along a choice of  $h$  simple closed loops  $\tau = \{\tau_1, \dots, \tau_h\}$ . If the length of each geodesic loop  $\tau_j$  is larger than  $2\pi$ ,  $M_\tau$  admits a hyperbolic structure [Thu78] and an analytic formula for  $\text{vol}(M_\tau)$  depending on the length of the  $\tau_j$  has been given in [NZ85].

**PROPOSITION 1.3.** *Let  $M_\tau$  be the compact 3-manifold obtained by Dehn filling from the hyperbolic 3-manifold  $M$ . If  $\rho_\tau : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  is the representation obtained from the composition of the quotient homomorphism  $\pi_1(M) \rightarrow \pi_1(M_\tau)$  with the holonomy representation of the hyperbolic structure on  $M_\tau$ , then*

$$\text{Vol}(\rho_\tau) = \text{vol}(M_\tau).$$

Thus with our cohomological definition,  $\text{Vol}(\rho)$  gives a continuous interpolation between the special values  $\text{vol}(M_\tau)$ . A natural question here is whether  $\text{Vol}$  is real analytic.

The structure of the paper is as follows. In § 2 we recall the basic properties of the Euler class and deduce from them the corresponding properties of the bounded Euler class. In § 3 we prove the vanishing of a “higher dimensional” rotation number for the image via the representation  $\rho$  of the fundamental groups of the toric cusps. In § 4 we relate the volume of the representation to the rotation numbers introduced in § 3 and thus prove the integrality of the values. In § 5 we relate the volume of the representation obtained via Dehn surgery to the volume of the corresponding hyperbolic manifold. Finally in the appendix we prove the continuity of the volume.

## 2. THE VOLUME VIEWED AS AN EULER CLASS AND THE HIRZEBRUCH PROPORTIONALITY PRINCIPLE IN BOUNDED COHOMOLOGY

Let  $\text{Isom}(\mathbb{H}^n)$  be the full group of isometries of real hyperbolic spaces  $\mathbb{H}^n$ . Its connected component of the identity consists of the group of orientation preserving isometries  $\text{Isom}^+(\mathbb{H}^n) = \text{Isom}(\mathbb{H}^n)^\circ$ . Let  $\epsilon : \text{Isom}(\mathbb{H}^n) \rightarrow \{-1, 1\}$  denote the

homomorphism with kernel  $\text{Isom}^+(\mathbb{H}^n)$ . We will need to consider  $\mathbb{R}$  as a non-trivial  $\text{Isom}(\mathbb{H}^n)$ -module via  $g_*t = \epsilon(g)t$ , for  $t \in \mathbb{R}$ , and denote by  $\mathbb{R}_\epsilon$  the resulting  $\text{Isom}(\mathbb{H}^n)$ -module.

Let  $\text{vol} : (\overline{\mathbb{H}^n})^{n+1} \rightarrow \mathbb{R}_\epsilon$  denote the signed volume of the convex hull of  $n+1$  points  $x_0, \dots, x_n$  in  $\overline{\mathbb{H}^n}$ : then  $\text{vol}$  is a  $\text{Isom}(\mathbb{H}^n)$ -equivariant bounded Borel cocycle whose restriction to  $(\mathbb{H}^n)^{n+1}$  is continuous. Hence it defines a continuous class  $\omega_n \in H_c^n(\text{Isom}(\mathbb{H}^n), \mathbb{R}_\epsilon)$  and a bounded continuous class  $\omega_n^b \in H_{cb}^n(\text{Isom}(\mathbb{H}^n), \mathbb{R}_\epsilon)$ , which correspond to each other via the isomorphism

$$(2.1) \quad c_{\mathbb{R}} : H_{cb}^n(\text{Isom}(\mathbb{H}^n), \mathbb{R}_\epsilon) \longrightarrow H_c^n(\text{Isom}(\mathbb{H}^n), \mathbb{R}_\epsilon)$$

(see [BBI13, Proposition 2]).

Assume that  $n = 2m$  is even and set  $G := \text{Isom}(\mathbb{H}^{2m})$ , which we identify with  $\text{SO}(2m, 1)$ . Since  $\text{SO}(2m)$  is the maximal compact in  $\text{SO}(2m, 1)^\circ$ , the corresponding classifying spaces are equal,  $\text{BSO}(2m) \cong \text{BSO}(2m, 1)^\circ$ . Thus the Euler class, being a singular cohomology class of  $\text{BSO}(2m)$  in degree  $2m$  (see [MS74, § 9]) is a singular cohomology class in  $H^{2m}(\text{BSO}(2m, 1)^\circ, \mathbb{Z})$ . Using Wigner's theorem [Wig73], according to which for any Lie group  $L$  the singular cohomology  $H^*(BL, \mathbb{Z})$  is isomorphic to the Borel cohomology<sup>1</sup>  $\widehat{H}_c^*(L, \mathbb{Z})$ , we obtain a class in  $\widehat{H}_c^{2m}(\text{SO}(2m, 1)^\circ, \mathbb{Z})$ . As the Euler class changes sign when the orientation is reversed, it extends to a class

$$\varepsilon_{2m} \in \widehat{H}_c^{2m}(G, \mathbb{Z}_\epsilon).$$

By Chern–Gauss–Bonnet theorem [Spi79, Chapter 13, Theorem 26] there is the correspondence

$$\begin{aligned} \widehat{H}_c^{2m}(G, \mathbb{Z}_\epsilon) &\longrightarrow H_c^{2m}(G, \mathbb{R}_\epsilon) \\ \varepsilon_{2m} &\longmapsto (-1)^m \frac{2}{\text{vol}(S^{2m})} \omega_{2m} \end{aligned}$$

under the change of coefficients from  $\mathbb{Z}_\epsilon$  to  $\mathbb{R}_\epsilon$ . The same assertion holds at the level of bounded cohomology, as we are going to establish:

**THEOREM 2.1.** *Let  $G := \text{Isom}(\mathbb{H}^{2m}) \cong \text{SO}(2m, 1)$ . The Euler class  $\varepsilon_{2m} \in \widehat{H}_c^{2m}(G, \mathbb{Z}_\epsilon)$  has a bounded representative  $\varepsilon_{2m}^b \in \widehat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon)$  that has the following properties:*

- (1) *it is unique, and*
- (2) *under the change of coefficients  $\mathbb{Z}_\epsilon \rightarrow \mathbb{R}_\epsilon$  it corresponds to*

$$(-1)^m \frac{2}{\text{vol}(S^{2m})} \omega_{2m}^b.$$

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<sup>1</sup>For definitions and basic facts concerning the Borel cohomology over  $\mathbb{Z}$  and  $\mathbb{R}$ , and its relation to continuous cohomology, see [BIW10, § 2.2 and § 2.3].

*Proof.* We consider the two long exact sequences, in ordinary and bounded cohomology, associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}_\epsilon \longrightarrow \mathbb{R}_\epsilon \longrightarrow (\mathbb{R}/\mathbb{Z})_\epsilon \longrightarrow 0,$$

namely

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\delta^b} & \hat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon) & \longrightarrow & H_{cb}^{2m}(G, \mathbb{R}_\epsilon) \longrightarrow H_c^{2m}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) \longrightarrow \dots \\ & & \parallel & & \downarrow c_{\mathbb{Z}} & & \cong \downarrow c_{\mathbb{R}} & & \parallel \\ \dots & \longrightarrow & H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\delta} & \hat{H}_c^{2m}(G, \mathbb{Z}_\epsilon) & \longrightarrow & H_c^{2m}(G, \mathbb{R}_\epsilon) \longrightarrow H_c^{2m}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) \longrightarrow \dots, \end{array}$$

where  $\delta$  and  $\delta^b$  are the connecting homomorphisms.

For ease of notation we set

$$\nu_{2m}^b := (-1)^m \frac{2}{\text{vol}(S^{2m})} \omega_{2m}^b \quad \text{and} \quad \nu_{2m} = (-1)^m \frac{2}{\text{vol}(S^{2m})} \omega_{2m}.$$

Since  $\nu_{2m}$  is the image of  $\varepsilon_{2m}$  under the change of coefficients  $\mathbb{Z}_\epsilon \rightarrow \mathbb{R}_\epsilon$ , we deduce that the image of  $\nu_{2m}$  in  $H_c^{2m}(G, (\mathbb{R}/\mathbb{Z})_\epsilon)$  vanishes and hence the same holds for  $\nu_{2m}^b$ . Thus there is  $\beta \in \hat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon)$  with image  $\nu_{2m}^b$ . But  $c_{\mathbb{Z}}(\beta) - \varepsilon_{2m}$  goes to zero in  $H_c^{2m}(G, \mathbb{R}_\epsilon)$ , hence  $c_{\mathbb{Z}}(\beta) - \varepsilon_{2m} = \delta(\eta)$  for some  $\eta \in H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon)$ . Thus  $c_{\mathbb{Z}}(\beta + \delta^b(\eta)) = \varepsilon_{2m}$ . This proves the second assertion in the theorem. The uniqueness follows from the following lemma.  $\square$

LEMMA 2.2. *The comparison map*

$$c_{\mathbb{Z}} : \hat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon) \longrightarrow \hat{H}_c^{2m}(G, \mathbb{Z}_\epsilon)$$

is injective.

*Proof.* We consider the following diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{cb}^{2m-1}(G, \mathbb{R}_\epsilon) & \longrightarrow & H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\delta^b} & \hat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon) \longrightarrow H_{cb}^{2m}(G, \mathbb{R}_\epsilon) \longrightarrow \dots \\ & & \downarrow c_{\mathbb{R}} & & \parallel & & \downarrow c_{\mathbb{Z}} & & \cong \downarrow c_{\mathbb{R}} \\ \dots & \longrightarrow & H_c^{2m-1}(G, \mathbb{R}_\epsilon) & \longrightarrow & H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\delta} & \hat{H}_c^{2m}(G, \mathbb{Z}_\epsilon) \longrightarrow H_c^{2m}(G, \mathbb{R}_\epsilon) \longrightarrow \dots \end{array}$$

Observe that  $H_c^{2m-1}(G, \mathbb{R}_\epsilon) = 0$ . Indeed this group injects into  $H_c^{2m-1}(G^\circ, \mathbb{R})$  by restriction, and the latter vanishes since there are no  $G^\circ$ -invariant differential  $(2m-1)$ -forms on  $\mathbb{H}^{2m}$ .

Let now  $\alpha \in \hat{H}_{cb}^{2m}(G, \mathbb{Z}_\epsilon)$  with  $c_{\mathbb{Z}}(\alpha) = 0$ . Since  $c_{\mathbb{R}}$  is an isomorphism (see (2.1)), we have that the image  $\alpha_{\mathbb{R}} \in H_{cb}^{2m}(G, \mathbb{R}_\epsilon)$  vanishes. Hence there is  $\beta \in H_c^{2m-1}(G, (\mathbb{R}/\mathbb{Z})_\epsilon)$  with  $\delta^b(\beta) = \alpha$ . But then  $\delta(\beta) = c_{\mathbb{Z}}(\alpha) = 0$ . Since  $H_c^{2m-1}(G, \mathbb{R}_\epsilon) = 0$ , by exactness  $\beta = 0$ , which finally implies that  $\alpha = 0$ .  $\square$

### 3. VANISHING OF “HIGHER ROTATION NUMBERS”

The bounded class  $\varepsilon_{2m}^b \in \widehat{H}_{cb}^{2m}(\mathrm{SO}(2m, 1), \mathbb{Z}_\epsilon)$  defined in the previous section restricts to a class on  $\mathrm{SO}(2m, 1)^\circ$  with trivial  $\mathbb{Z}$ -coefficients. When  $m = 1$  and

$$\rho : \mathbb{Z} \rightarrow \mathrm{SO}(2, 1)^\circ$$

is a homomorphism, then  $\rho^*(\varepsilon_2^b) \in H_b^2(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$  is the rotation number of  $\rho(1)$ , [Ghy87]. In contrast to this, in higher dimension we have the following:

**THEOREM 3.1.** *Let  $m \geq 2$  and let  $\rho : \mathbb{Z}^{2m-1} \rightarrow \mathrm{SO}(2m, 1)^\circ$  be a homomorphism. Then  $\rho^*(\varepsilon_{2m}^b)$  vanishes in  $H_b^{2m}(\mathbb{Z}^{2m-1}, \mathbb{Z})$ .*

Before proving the theorem we need some information about Abelian subgroups of  $\mathrm{SO}(2m, 1)^\circ$ . To this purpose, recall that

$$\mathrm{SO}(2m, 1) := \left\{ A \in \mathrm{GL}(2m+1, \mathbb{R}) : \det A = 1, A \text{ preserves } q(x) := \sum_{i=1}^{2m} x_i^2 - x_{2m+1}^2 \right\}.$$

Then the maximal compact subgroup  $K < \mathrm{SO}(2m, 1)$  is the image of  $\mathrm{O}(2m)$  under the homomorphism

$$\begin{aligned} \mathrm{O}(2m) &\longrightarrow \mathrm{SO}(2m, 1) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}, \end{aligned}$$

and the image  $K^\circ$  of  $\mathrm{SO}(2m)$  is the maximal compact subgroup of  $\mathrm{SO}(2m, 1)^\circ$ . If  $T$  is the image of

$$\begin{aligned} \mathrm{O}(2)^m &\longrightarrow \mathrm{SO}(2m, 1) \\ (A_1, \dots, A_m) &\longmapsto \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_m & \\ & & & \prod_{i=1}^m \det A_i \end{pmatrix}, \end{aligned}$$

we define

$$T_0 := T \cap K^\circ.$$

Now  $\mathrm{SO}(2m, 1)^\circ$  preserves each connected component of the two-sheeted hyperboloid

$$x_1^2 + \dots + x_{2m}^2 - x_{2m+1}^2 = -1.$$

As a result, the parabolic subgroup  $P = \text{Stab}_{\text{SO}(2m,1)^\circ}(\mathbb{R}(e_1 - e_{2m+1}))$  admits the decomposition  $P = MAN$ , where

$$M := \left\{ m(U) := \begin{pmatrix} 1 & & \\ & U & \\ & & 1 \end{pmatrix} : U \in \text{SO}(2m-1) \right\}$$

$$A := \left\{ a(t) := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \text{Id} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}$$

and

$$N := \left\{ n(x) := \begin{pmatrix} 1 - \frac{\|x\|^2}{2} & -x & -\frac{\|x\|^2}{2} \\ {}^t x & I & {}^t x \\ \frac{\|x\|^2}{2} & x & 1 + \frac{\|x\|^2}{2} \end{pmatrix} : x \in \mathbb{R}^{2m-1} \right\}.$$

LEMMA 3.2. *Let  $B < \text{SO}(2m,1)^\circ$  be an Abelian group. Then up to conjugation one of the following holds:*

- (1)  $B < P$ ;
- (2)  $B < T_0$ .

*Proof.* We argue geometrically by considering the action of  $B$  on  $\overline{\mathbb{H}^{2m}}$ . One of the following holds:

- (i)  $B$  fixes a point in  $\partial\mathbb{H}^{2m}$  and thus we have  $B < P$  up to conjugation;
- (ii)  $B$  fixes a point in  $\mathbb{H}^{2m}$  and hence  $B < K^\circ \cong \text{SO}(2m)$  up to conjugation. Since  $B$  is Abelian, it can be simultaneously reduced to a diagonal  $2 \times 2$  bloc form and hence can be conjugated into  $T_0$ .
- (iii) There is a geodesic  $g \in \mathbb{H}^{2m}$  that is left globally invariant by  $B$ . If  $\{g_-, g_+\}$  are its endpoints, then either  $Bg_+ = g_+$  and we are in case (i) above, or there is  $b \in B$  with  $bg_+ = g_-$ . But then  $B$  fixes the unique  $b$ -fixed point  $g_0 \in g$ , hence we are in case (ii).

□

LEMMA 3.3. *Let  $P < \text{SO}(2m,1)^\circ$  be the minimal parabolic subgroup as above. Then:*

$$H_c^{2m}(P, \mathbb{R}) = 0 \quad \text{and} \quad H_c^{2m-1}(P, \mathbb{R}) = 0.$$

*Proof.* We use Mostow's theorem [BW00] and compute the cohomology of the complex of  $P$ -invariant forms on  $M \setminus P$  in the relevant degrees. Since  $\dim(M \setminus P) = 2m$ , the space of  $P$ -invariant  $2m$ -forms is one dimensional; both assertions in the lemma will follow if we show that the space of  $P$ -invariant  $(2m-1)$ -forms is one dimensional and that on this space, the exterior derivative does not vanish.

To this end, we identify

$$\begin{aligned} M \setminus P &\longrightarrow \mathbb{R} \times \mathbb{R}^{2m-1} \\ Ma(t)n(x) &\longmapsto (t, x) \end{aligned}$$



and in these coordinates the right translation  $R_p$  by an element  $p = m(U)a(s)n(y)$  is given by

$$R_p(t, x) = (t + s, U^{-1}s^2x + y).$$

Thus invariant  $(2m - 1)$ -forms on  $M \setminus P$  correspond to  $(2m - 1)$ -multilinear anti-symmetric forms on  $\mathbb{R} \times \mathbb{R}^{2m-1}$  that are  $(\text{Id} \times \text{SO}(2m - 1))$ -invariant; by duality with the space of linear forms, this space is one dimensional and generated by

$$((t_1, x_1), \dots, (t_{2m-1}, x_{2m-1})) \mapsto \det(x_1, \dots, x_{2m-1}).$$

The corresponding invariant  $(2m - 1)$ -form is then given by

$$\omega = e^{-(2m-1)t} dx_1 \wedge \dots \wedge dx_{2m-1}$$

whose differential is

$$d\omega = -(2m - 1)e^{-(2m-1)t} dt \wedge dx_1 \wedge \dots \wedge dx_{2m-1}$$

and thus is not vanishing.  $\square$

LEMMA 3.4. *If  $m \geq 2$  then the restriction*

$$\varepsilon_{2m}^b|_P \in \hat{H}_b^{2m}(P, \mathbb{Z})$$

*vanishes.*

*Proof.* Considering the long exact sequence in bounded and ordinary cohomology associated to

$$(3.1) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

and taking into account Lemma 3.4 and the vanishing of bounded real cohomology of  $P$ , we obtain the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_c^{2m-1}(P, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & \hat{H}_{cb}^{2m}(P, \mathbb{Z}) & \longrightarrow & 0 \\ & & \parallel & & \downarrow c_{\mathbb{Z}} & & \\ 0 & \longrightarrow & H_c^{2m-1}(P, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta} & \hat{H}_c^{2m}(P, \mathbb{Z}) & \longrightarrow & 0. \end{array}$$

Thus  $\hat{H}_{cb}^{2m}(P, \mathbb{Z}) \cong \hat{H}_c^{2m}(P, \mathbb{Z})$ . By Wigner's theorem [Wig73], the restriction to  $M$  induces an isomorphism  $\hat{H}_c^{2m}(P, \mathbb{Z}) \cong \hat{H}_c^{2m}(M, \mathbb{Z})$ . Since however the Euler class  $\varepsilon_{2m}$  of  $\text{SO}(2m)$  restricted to  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} : U \in \text{SO}(2m - 1) \right\}$  vanishes, we conclude that  $\varepsilon_{2m}|_P = 0$  and hence  $\varepsilon_{2m}^b|_P = 0$ .  $\square$

We deduce then using Lemma 3.4 that Theorem 3.1 holds if the image of  $\rho$  is contained in  $P$ . We must therefore turn to the case in which  $\rho(\mathbb{Z}^{2m-1})$  lies in  $T_0$ . Let then

$$\pi_i : T \longrightarrow \text{O}(2)$$

be the projection on the  $i$ -th factor of  $T$  and let

$$\varepsilon_{(i)} := \pi_i^*(\varepsilon_2) \quad \text{and} \quad \varepsilon_{(i)}^b := \pi_i^*(\varepsilon_2^b),$$

where  $\varepsilon_2$  and  $\varepsilon_2^b$  are respectively the Euler class and the bounded Euler class of  $O(2)$ . Observe that if  $L$  is compact, the long exact sequence above gives

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H_c^\bullet(L, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\cong} & \widehat{H}_{cb}^\bullet(L, \mathbb{Z}_\epsilon) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & H_c^\bullet(L, (\mathbb{R}/\mathbb{Z})_\epsilon) & \xrightarrow{\cong} & \widehat{H}_c^\bullet(L, \mathbb{Z}_\epsilon) & \longrightarrow & 0. \end{array}$$

Since the ordinary Euler class is a characteristic class, we have

$$(3.3) \quad \varepsilon_{2m}|_T = \varepsilon_{(1)} \cup \cdots \cup \varepsilon_{(m)},$$

and hence it follows from (3.2) that

$$(3.4) \quad \varepsilon_{2m}^b|_T = \varepsilon_{(1)}^b \cup \cdots \cup \varepsilon_{(m)}^b.$$

Set  $T^0$  to be the image of  $SO(2)^m$  in  $T$ .

LEMMA 3.5. *If  $\rho : \mathbb{Z}^{2m-1} \rightarrow T_0$  does not take values in  $T^0$ , then  $\rho^*(\varepsilon_{2m}^b) = 0$ .*

*Proof.* An Abelian subgroup of  $O(2)$  not contained in  $SO(2)$  is either trivial or of the form  $\{1, \sigma\}$ , with  $\sigma^2 = e$ . Thus if  $\rho(\mathbb{Z}^{2m-1}) \not\subset T^0$ , there is  $\pi_i$  such that  $\pi_i \rho(\mathbb{Z}^{2m-1}) \subset \{1, \sigma\}$ . But since  $\varepsilon_2^b$  vanishes on such a subgroup, we get  $\rho^*(\varepsilon_{2m}^b) = 0$  by using (3.4).  $\square$

Thus we are reduced to analyze homomorphisms

$$\rho : \mathbb{Z}^{2m-1} \longrightarrow T^0 \cong SO(2)^m.$$

As before, since  $SO(2)^m$  is compact, the connecting homomorphism

$$\delta^b : H_c^{2m-1}(SO(2)^m, \mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{H}_{cb}^{2m}(SO(2)^m, \mathbb{Z})$$

is an isomorphism. For  $m = 1$  let

$$\begin{aligned} \text{Rot} : SO(2) \times SO(2) &\longrightarrow SO(2) \cong \mathbb{R}/\mathbb{Z} \\ (g, h) &\longmapsto g^{-1}h \end{aligned}$$

denote the homogeneous 1-cocycle, where we fix the orientation preserving identification

$$SO(2) \longrightarrow \mathbb{R}/\mathbb{Z},$$

and

$$[\text{Rot}] \in H_c^1(SO(2), \mathbb{R}/\mathbb{Z})$$

the corresponding class. Then  $\delta^b([\text{Rot}]) = \varepsilon_2^b$  and from this we deduce that if

$$\vartheta = \pi_1^*([\text{Rot}]) \in H_c^1(SO(2)^m, \mathbb{R}/\mathbb{Z}),$$

then  $\delta^b$  maps the class  $\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)} \in H_c^{2m-1}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z})$  to the class  $\varepsilon_{(1)}^b \cup \cdots \cup \varepsilon_{(m)}^b \in \widehat{H}_{\mathrm{cb}}^{2m}(\mathrm{SO}^m, \mathbb{Z})$ . As a result, it follows from the commutativity of the square

$$\begin{array}{ccc} H_c^{2m-1}(\mathrm{SO}(2)^m, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & \widehat{H}_{\mathrm{cb}}^{2m}(\mathrm{SO}(2)^m, \mathbb{Z}) \\ \rho^* \downarrow & & \downarrow \rho^* \\ H^{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & H_b^{2m}(\mathbb{Z}^{2m-1}, \mathbb{Z}) \end{array}$$

that

$$(3.5) \quad \rho^*(\varepsilon_{2m}^b) = \delta^b(\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)})).$$

In order to prove the vanishing of the left hand side, we are going to show that  $\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \cdots \cup \varepsilon_{(m)}) = 0$ . To this end we use that the pairing

$$(3.6) \quad \begin{array}{ccc} H^{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ \beta & \longmapsto & \langle \beta, [\mathbb{Z}^{2m-1}] \rangle \end{array}$$

is an isomorphism, where  $[\mathbb{Z}^{2m-1}] \in H_{2m-1}(\mathbb{Z}^{2m-1}, \mathbb{Z}) \cong \mathbb{Z}$  denotes the fundamental class.

We will need the following

**LEMMA 3.6.** *Let  $n \geq 1$  and let  $e_1, \dots, e_n$  be the canonical basis of  $\mathbb{Z}^n$ . Then the group chain*

$$z = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sign}(\sigma) [0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} + \cdots + e_{\sigma(n)}]$$

*is a representative of the fundamental class  $[\mathbb{Z}^n] \in H_n(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}$ .*

*Proof.* It is a straightforward computation to check that  $\partial z = 0$  so that  $z$  is indeed a cycle. Let  $\omega_{\mathbb{R}^n} \in H^n(\mathbb{Z}^n, \mathbb{R})$  be the Euclidean volume class. Note that the volume class evaluates to 1 on the fundamental class, as the  $n$ -torus generated by the canonical basis has volume 1. A cocycle representing  $\omega_{\mathbb{R}^n}$  is given by  $V : (\mathbb{Z}^n)^{n+1} \rightarrow \mathbb{R}$  sending  $v_0, v_1, \dots, v_n$  to the signed volume of the simplex with the vertices in the lattice  $\mathbb{Z}^n \subset \mathbb{R}^n$ , that is

$$V(v_0, v_1, \dots, v_n) = \frac{1}{n!} \det(v_1 - v_0, \dots, v_n - v_0).$$

In order to compute  $\langle \omega_{\mathbb{R}^n}, [\mathbb{Z}^n] \rangle = V(z)$  we need to evaluate  $V$  on each summand of  $z$ . By definition,

$$\begin{aligned} & V(0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} + \dots + e_{\sigma(n)}) \\ &= \frac{1}{n!} \det(e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \dots, e_{\sigma(1)} + \dots + e_{\sigma(n)}) \\ &= \text{sign}(\sigma) \frac{1}{n!} \det(e_1, e_1 + e_2, \dots, e_1 + \dots + e_n) \\ &= \text{sign}(\sigma) \frac{1}{n!}, \end{aligned}$$

where we have used for the second equality the fact that the determinant is alternating with respect to line permutations. Summing up, we get

$$\langle \omega_{\mathbb{R}^n}, [z] \rangle = \sum_{\sigma \in \text{Sym}(n)} \frac{1}{n!} = 1,$$

thus proving the lemma.  $\square$

LEMMA 3.7. *Let  $m \geq 2$ . Then*

$$\langle \rho^*(\vartheta \cup \varepsilon_{(2)} \cup \dots \cup \varepsilon_{(m)}), [\mathbb{Z}^{2m-1}] \rangle = 0.$$

*Proof.* We use as a representative of  $\varepsilon_2 \in H^2(\text{SO}(2), \mathbb{Z})$  the multiple  $-\frac{1}{2}$  of the orientation cocycle. The cocycle takes values in  $\frac{1}{2}\mathbb{Z}$  but represents an integral cocycle and in particular evaluates to an integer on a fundamental class. Hence a representative for  $\kappa := \vartheta \cup \varepsilon_{(2)} \cup \dots \cup \varepsilon_{(m)} \in H_c^{2m-1}((\text{SO}(2))^m, \mathbb{R}/\mathbb{Z})$  is given by the cocycle mapping  $g_0, \dots, g_{2m-1} \in (\text{SO}(2))^m$  to the product

$$\frac{(-1)^{m-1}}{2^{m-1}} \cdot \text{Rot}_1(g_0, g_1) \cdot \text{Or}_2(g_1, g_2, g_3) \cdots \text{Or}_m(g_{2m-3}, g_{2m-2}, g_{2m-1}),$$

where  $\text{Rot}_1$  denotes the pullback to  $\text{SO}(2)^m$  via the first projection  $\pi_1$  of the homogeneous 1-cocycle  $\text{Rot}$  defined above, while  $\text{Or}_j$  denotes the pullback to  $\text{SO}(2)^m$  via  $\pi_j$  of the orientation cocycle

$$\text{Or} : \text{SO}(2)^3 \longrightarrow \{-1, 0, 1\}.$$

We now evaluate the pullback on the cycle  $z$  of Lemma 3.6 and, writing  $f_i = \rho(e_i) \in (\text{SO}(2))^m$ , obtain

$$\begin{aligned} \langle \rho^*(\kappa), [\mathbb{Z}^{2m-1}] \rangle &= \frac{(-1)^{m-1}}{2^{m-1}} \sum_{\sigma \in \text{Sym}(n)} \text{sign}(\sigma) \text{Rot}_1(0, f_{\sigma(1)}) \\ &\quad \cdot \text{Or}_2(f_{\sigma(1)}, f_{\sigma(1)} f_{\sigma(2)}, f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}) \\ &\quad \cdot (\text{Or}_3 \cup \dots \cup \text{Or}_m)(f_{\sigma(1)} f_{\sigma(2)} f_{\sigma(3)}, \dots, f_{\sigma(1)} \cdots f_{\sigma(n)}). \end{aligned}$$

If instead of summing over  $\sigma \in \text{Sym}(n)$  we add also the permutations  $\sigma \circ (2\ 3)$ , then every permutation is counted twice and the only difference between  $\sigma$  and  $\sigma \circ (2\ 3)$  appears in the second factor of the summand, where instead of

$\text{Or}_2(f_{\sigma(1)}, f_{\sigma(1)}f_{\sigma(2)}, f_{\sigma(1)}f_{\sigma(2)}f_{\sigma(3)})$  one would get  $\text{Or}_2(f_{\sigma(1)}, f_{\sigma(1)}f_{\sigma(3)}, f_{\sigma(1)}f_{\sigma(2)}f_{\sigma(3)})$  (where we used that  $\text{SO}(2)$  is Abelian). Thus, the above sum can be rewritten as

$$\begin{aligned} \langle \rho^*(\kappa), [\mathbb{Z}^{2m-1}] \rangle &= \frac{(-1)^{m-1}}{2^m} \sum_{\sigma \in \text{Sym}(n)} \text{sign}(\sigma) \text{Rot}_1(0, f_{\sigma(1)}) \\ &\cdot (\text{Or}_2(f_{\sigma(1)}, f_{\sigma(1)}f_{\sigma(2)}, f_{\sigma(1)}f_{\sigma(2)}f_{\sigma(3)}) - \text{Or}_2(f_{\sigma(1)}, f_{\sigma(1)}f_{\sigma(3)}, f_{\sigma(1)}f_{\sigma(2)}f_{\sigma(3)})) \\ &\cdot (\text{Or}_3 \cup \dots \cup \text{Or}_m)(f_{\sigma(1)}f_{\sigma(2)}f_{\sigma(3)}, \dots, f_{\sigma(1)} \dots f_{\sigma(n)}). \end{aligned}$$

We will now show that the middle factor of each summand vanishes for every  $\sigma \in \text{Sym}(n)$ . To simplify the notation we can by symmetry suppose that  $\sigma(1) = 1$ ,  $\sigma(2) = 2$  and  $\sigma(3) = 3$ . Using that  $\text{Or}_2$  is  $(\text{SO}(2))^m$ -invariant we obtain

$$\text{Or}_2(f_1, f_1f_2, f_1f_2f_3) - \text{Or}_2(f_1, f_1f_3, f_1f_2f_3) = \text{Or}_2(f_2^{-1}, \text{id}, f_3) - \text{Or}_2(f_3^{-1}, \text{id}, f_2).$$

The right hand side of the expression is equal to 0 since the orientation cocycle  $\text{Or}_2$  is alternating and the equality

$$\text{Or}_2(f, g, h) = -\text{Or}_2(f^{-1}, g^{-1}, h^{-1}),$$

holds for any  $f, g, h$  in  $(\text{SO}(2))^m$ .  $\square$

*Proof of Theorem 3.1.* Let  $\rho : \mathbb{Z}^{2m-1} \rightarrow \text{SO}(2m, 1)^\circ$  be a homomorphism. By Lemma 3.2 either  $\rho(\mathbb{Z}^{2m-1}) < P$  up to conjugation and then  $\rho^*(\varepsilon_{2m}^b) = 0$  by Lemma 3.4 or  $\rho^*(\mathbb{Z}^{2m-1}) < T_0$  up to conjugacy. Then either  $\rho(\mathbb{Z}^{2m-1}) \not\subset T^\circ$  and the vanishing follows from Lemma 3.5 or  $\rho(\mathbb{Z}^{2m-1}) < T^\circ$ , in which case Lemma 3.7 and (3.5) imply that  $\rho^*(\vartheta \cup \varepsilon_{(2)} \cup \dots \cup \varepsilon_{(m)}) = 0$  and hence  $\rho^*(\varepsilon_{2m}^b) = 0$  by (3.6).  $\square$

#### 4. CONGRUENCE RELATIONS FOR VOLUMES

Let  $M$  be a complete finite volume hyperbolic  $n$ -dimensional manifold and let  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^n)$  be a homomorphism. Given a compact core  $N$  of  $M$  we consider  $\rho$  as a representation of  $\pi_1(N)$  and use the pullback via  $\rho$  in bounded cohomology together with the isomorphism

$$H_b^n(\pi_1(N), \mathbb{R}) \cong H_b^n(N, \mathbb{R})$$

to obtain a bounded singular class in  $H_b^n(N, \mathbb{R})$ , denoted  $\rho^*(\omega_n^b)$  by abuse of notation. If  $n = 2m$ , by the same abuse of notation, and by considering again the pullback in bounded cohomology via  $\rho$ , this time together with the isomorphism

$$H_b^{2m}(\pi_1(N), \mathbb{Z}) \cong H_b^{2m}(N, \mathbb{Z}),$$

we obtain a class  $\rho^*(\varepsilon_{2m}^b) \in H_b^{2m}(N, \mathbb{Z})$ , which is a bounded singular integral class. Recall that the volume of  $\rho$  is defined using the isometric isomorphism

$$(4.1) \quad j : H_b^{2m}(N, \partial N, \mathbb{R}) \longrightarrow H_b^{2m}(N, \mathbb{R})$$

by

$$\text{Vol}(\rho) := \langle j^{-1}(\rho^*(\omega_{2m}^b)), [N, \partial N] \rangle.$$

Finally, denoting  $\delta^b$  the connecting homomorphism in the long exact sequence in bounded singular cohomology

$$(4.2) \quad \delta^b : H^{2m-1}(\partial N, \mathbb{R}/\mathbb{Z}) \longrightarrow H_b^{2m}(\partial N, \mathbb{Z})$$

(which is in fact an isomorphism), we have:

**THEOREM 4.1.** *Let  $M$  be a complete hyperbolic manifold of finite volume and even dimension  $n = 2m$  and  $N$  a compact core of  $M$ . If  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^{2m})$ , then*

$$\frac{-2}{\text{vol}(S^{2m})} \cdot \text{Vol}(\rho) \equiv -\langle (\delta^b)^{-1} \rho^*(\varepsilon_{2m}^b)|_{\partial N}, [\partial N] \rangle \pmod{\mathbb{Z}}.$$

*Proof.* In fact, (4.1) and (4.2) are part of the following big diagram

$$\begin{array}{ccccc} H^{2m-1}(\partial N, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & H_b^{2m}(\partial N, \mathbb{Z}) & \longrightarrow & H_b^{2m}(\partial N, \mathbb{R}) = 0 \\ & & \uparrow & & \uparrow \\ & & H_b^{2m}(N, \mathbb{Z}) & \longrightarrow & H_b^{2m}(N, \mathbb{R}) \\ & & & & \uparrow j \\ & & & & H_b^{2m}(N, \partial N, \mathbb{R}), \end{array}$$

where the rows are obtained from the long exact sequence induced by the change of coefficients in (3.1) and from the fact that  $H_b^\bullet(\partial N, \mathbb{R}) = 0$  since  $\pi_1(\partial N)$  is amenable; the columns on the other hand follow from the long exact sequence in relative bounded cohomology associated to the inclusion of pairs  $(N, \emptyset) \hookrightarrow (N, \partial N)$  (see [BIW10, § 2.2]). Let  $z$  be a  $\mathbb{Z}$ -valued singular bounded cocycle representing  $\rho^*(\varepsilon_{2m}^b) \in H_b^{2m}(N, \mathbb{Z})$ . Restricting  $z$  to the boundary  $\partial N$  we obtain a  $\mathbb{Z}$ -valued singular bounded cocycle  $z|_{\partial N}$ , which we know is a coboundary when considered as a  $\mathbb{R}$ -valued cocycle since  $H_b^{2m}(\partial N, \mathbb{R}) = 0$ . Thus there must exist a bounded  $\mathbb{R}$ -valued singular  $(2m-1)$ -cochain  $b$  on  $\partial N$  such that

$$(z|_{\partial N})_{\mathbb{R}} = db,$$

where  $d$  is the differential operator on (bounded  $\mathbb{R}$ -valued) singular cochains.

On the one hand, we note that since  $db$  is  $\mathbb{Z}$ -valued, the cochain  $b \pmod{\mathbb{Z}}$  is a  $\mathbb{R}/\mathbb{Z}$ -valued cocycle on  $\partial N$  whose cohomology class is mapped to

$$[\rho^*(\varepsilon_{2m}^b)|_{\partial N}] = [z|_{\partial N}] = \delta^b([b \pmod{\mathbb{Z}}])$$

by the connecting homomorphism  $\delta^b$  in (4.2).

On the other hand, define a bounded  $\mathbb{R}$ -valued singular cochain  $\bar{b}$  on  $N$  by

$$\bar{b}(\sigma) := \begin{cases} b(\sigma) & \text{if } \sigma \subset \partial N, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $[z_{\mathbb{R}} - d\bar{b}] = [z_{\mathbb{R}}] \in H_b^{2m}(N, \mathbb{R})$ , and since  $z_{\mathbb{R}} - d\bar{b}$  vanishes on  $\partial N$  we have actually constructed a cocycle representing the relative bounded class  $j^{-1}(\rho^*(\varepsilon_{2m}^b)) \in H_b^{2m}(N, \partial N, \mathbb{R})$ .

It remains to evaluate  $j^{-1}(\rho^*(\varepsilon_{2m,\mathbb{R}}^b))$  on  $[N, \partial N]$  by using this specific cocycle. Let  $t$  be a singular chain obtained from a triangulation of  $N$  that induces a triangulation of  $\partial N$ , so that  $t$  represents the relative fundamental class  $[N, \partial N]$  over  $\mathbb{Z}$ . Then we obtain

$$\begin{aligned} \langle j^{-1}(\rho^*(\varepsilon_{2m})), [N, \partial N] \rangle \mod \mathbb{Z} &= \langle z_{\mathbb{R}} - d\bar{b}, t \rangle \mod \mathbb{Z} \\ &= \underbrace{\langle z_{\mathbb{R}}, t \rangle}_{\in \mathbb{Z}} - \langle \bar{b}, \partial t \rangle \mod \mathbb{Z} \\ &= -\langle b \mod \mathbb{Z}, [\partial N] \rangle \\ &= -\langle (\delta^b)^{-1}(\rho^*(\varepsilon_{2m})|_{\partial N}), [\partial N] \rangle. \end{aligned}$$

□

*Proof of Theorem 1.1.* Let  $M$  be complete, of finite volume and with cusp cross sections  $C_1, \dots, C_h$ . Then it follows from Theorem 4.1 that

$$\frac{-2}{\text{vol}(S^{2m})} \text{Vol}(\rho) \equiv - \sum_{i=1}^h \langle (\delta^b)^{-1}(\rho^*(\varepsilon_{2m})|_{C_i}), [C_i] \rangle \mod \mathbb{Z},$$

with the usual abuse of notation that  $\rho^*(\varepsilon_{2m}^b)|_{C_i}$  refers to the element in  $H_b^{2m}(C_i, \mathbb{Z})$  corresponding to  $(\rho|_{\pi_1(C_i)})^*(\varepsilon_{2m}^b) \in H_b^{2m}(\pi_1(C_i), \mathbb{Z})$ .

If now all the  $C_i$ 's are tori, the above congruence relation and Theorem 3.1 imply that  $\frac{-2}{\text{vol}(S^{2m})} \text{Vol}(\rho) \in \mathbb{Z}$ .

In the general case, let  $p_i : C'_i \rightarrow C_i$  be a covering of degree  $B_{2m-1}$  that is a torus. Then

$$\begin{aligned} B_{2m-1} \langle (\delta^b)^{-1}(\rho^*(\varepsilon_{2m}^b)|_{C_i}), [C_i] \rangle &= \langle (\delta^b)^{-1}(\rho^*(\varepsilon_{2m}^b)|_{C_i}), p_{i*}([C'_i]) \rangle \\ &= \langle (\delta^b)^{-1}p_i^*(\rho^*(\varepsilon_{2m}^b)|_{C_i}), [C'_i] \rangle. \end{aligned}$$

Now observe that  $p_i^*(\rho^*(\varepsilon_{2m}^b)|_{C_i}) \in H_b^{2m}(C'_i, \mathbb{Z})$  corresponds to the class

$$(\rho \circ (p_i)_*)^*(\varepsilon_{2m}^b) = (\rho|_{\pi_1(C'_i)})^*(\varepsilon_{2m}^b) \in H_b^{2m}(\pi_1(C'_i), \mathbb{Z}),$$

which vanishes by Theorem 3.1.

The case when  $\Gamma$  has torsion is immediate. □

## 5. EXAMPLES OF NONTRIVIAL AND NON MAXIMAL REPRESENTATIONS

**5.1. Dimension 3: representations given by Dehn filling.** Let  $M$  be a complete finite volume hyperbolic 3-manifold, which, for simplicity, we assume has only one cusp. If  $N$  is a compact core of  $M$ , its boundary  $\partial N$  is Euclidean with the induced metric and hence there is an isometry  $\varphi : \partial N \rightarrow \mathbb{T}^2$  to a two-dimensional torus for an appropriate flat metric on  $\mathbb{T}^2$ . We obtain then a decomposition of  $M$  as a connected sum

$$M = N \# (\mathbb{T}^2 \times \mathbb{R}_{\geq 0}),$$

where the identification is via  $\varphi$ . We are now going to fill in a solid two-torus to obtain a compact manifold. To this end, let  $\tau \subset \partial N$  be a simple closed geodesic and let us choose a diffeomorphism  $\varphi_\tau : \partial N \rightarrow S^1 \times S^1$ , in such a way that  $\varphi_\tau(\tau) = S^1 \times \{*\}$ . Then  $M_\tau$  is the connected sum

$$M_\tau := N \# (\mathbb{D}^2 \times S^1),$$

identified via  $\varphi_\tau$ .

Denote by  $j_\tau : N \hookrightarrow M_\tau$  the canonical inclusion and by  $p : M \rightarrow N$  the canonical projection given by the cusp retraction  $\mathbb{T}^2 \times \mathbb{R}_{>0} \rightarrow \mathbb{T}^2$ . The composition

$$f_\tau = j_\tau \circ p : M \longrightarrow M_\tau$$

induces a map

$$(f_\tau)_* : \Gamma \longrightarrow \Gamma_\tau$$

of the fundamental groups  $\Gamma = \pi_1(M)$  and  $\Gamma_\tau = \pi_1(M_\tau)$ .

**PROPOSITION 5.1.** *Let  $M_\tau$  be the compact 3-manifold obtained by Dehn filling from the hyperbolic 3-manifold  $M$  with one cusp. Let  $\rho : \Gamma_\tau \rightarrow \mathrm{SO}(3, 1)$  be any representation of  $\Gamma_\tau$  and let  $\rho_\tau := \rho \circ f_\tau : \Gamma \rightarrow \mathrm{SO}(3, 1)$ . Then*

$$\mathrm{Vol}(\rho_\tau) = \mathrm{Vol}(\rho).$$

By Gromov–Thurston  $2\pi$ -Theorem [GT87], for all geodesic curves  $\tau$  for which the induced length is greater than  $2\pi$  in the induced Euclidean metric on  $\partial N$ , the compact manifold  $M_\tau$  admits a hyperbolic structure. Proposition 1.3 is then an immediate consequence of Proposition 5.1.

To prove the proposition, recall that by definition, the volume of the representation  $\rho_\tau$  is equal to

$$\mathrm{Vol}(\rho_\tau) = \langle c \circ \Psi^{-1} \circ f^* \circ \rho^*(\omega_3^b), [N, \partial N] \rangle,$$

where all maps involved can be read in the diagram below. We will start by defining a map  $F : H^3(M_\tau) \rightarrow H^3(N, \partial N)$  that will turn the diagram below into a commutative diagram (Lemma 5.3) and which will induce a canonical isomorphism (Lemma 5.2).

$$(5.1) \quad \begin{array}{ccccccc} & & \xrightarrow{\rho_\tau^*} & & & & \\ & & \text{---} & & \text{---} & & \\ H_{\mathrm{cb}}^3(\mathrm{SO}(3, 1)) & \xrightarrow{\rho^*} & H_b^3(\Gamma_\tau) & \xrightarrow{f_\tau^*} & H_b^3(\Gamma) & \xleftarrow[\Psi]{\cong} & H_b^3(N, \partial N) \\ \downarrow c & & \downarrow c & & & & \downarrow c \\ H_c^3(\mathrm{SO}(3, 1)) & \xrightarrow{\rho^*} & H^3(\Gamma_\tau) & \xrightarrow[\cong]{g} & H^3(M_\tau) & \xrightarrow[\text{---}]{F} & H^3(N, \partial N). \end{array}$$



The inclusions

$$\begin{array}{ccc}
 M_\tau & & (N, \partial N) \\
 & \searrow i & \swarrow (j_\tau, \varphi_\tau) \\
 & (M_\tau, \mathbb{D}^2 \times S^1) &
 \end{array}$$

induce the following homology and cohomology maps

$$(5.2) \quad \begin{array}{ccc}
 H_\bullet(M_\tau, \mathbb{Z}) & & H_\bullet((N, \partial N), \mathbb{Z}) \\
 & \searrow i_* & \swarrow (j_\tau, \varphi_\tau)_* \\
 & H_\bullet((M_\tau, \mathbb{D}^2 \times S^1), \mathbb{Z}) &
 \end{array}$$

and

$$(5.3) \quad \begin{array}{ccc}
 H^\bullet(M_\tau, \mathbb{Z}) & & H^\bullet((N, \partial N), \mathbb{Z}) \\
 & \swarrow i^* & \searrow (j_\tau, \varphi_\tau)^* \\
 & H^\bullet((M_\tau, \mathbb{D}^2 \times S^1), \mathbb{Z}) &
 \end{array}$$

LEMMA 5.2. *In degree 3 the maps in (5.2) and (5.3) are canonical isomorphisms and the composition*

$$F = (j_\tau, \varphi_\tau)^* \circ (i^*)^{-1} : H^3(M_\tau, \mathbb{Z}) \longrightarrow H^3(N, \partial N, \mathbb{Z})$$

*maps the dual  $\beta_{M_\tau}$  of the fundamental class of  $M_\tau$  to the dual  $\beta_{[N, \partial N]}$  of the fundamental class of  $(N, \partial N)$ .*

*Proof.* It is enough to show the statement in homology where we show that fundamental classes are mapped to each other by showing the existence of a compatible triangulation of the three manifolds. Start with a triangulation of the boundary torus  $S^1 \times S^1 = \partial N$ , extend it on the one hand to the filled torus  $\mathbb{D}^2 \times S^1$  and on the other hand to  $N$ , [Mun66]. This produces compatible triangulations representing  $[M_\tau]$ ,  $[M_\tau, \mathbb{D}^2 \times S^1]$  and  $[N, \partial N]$ .  $\square$

LEMMA 5.3. *The diagram (5.1) commutes.*

*Proof.* We only need to show that the right rectangle commutes. For this, we will decompose the diagram in subdiagrams as follows:

$$\begin{array}{ccccc}
 & & \xrightarrow{\quad \Psi \quad} & & \\
 & \xrightarrow{\quad \cong \quad} & H_b^3(M) & \xleftarrow[p^*]{\quad \cong \quad} & H_b^3(N) \\
 & \uparrow f^* & \uparrow f^* & \nearrow j_\tau^* & \nwarrow i_{|N}^* \\
 H_b^3(\Gamma) & \xrightarrow{\quad \cong \quad} & H_b^3(M_\tau) & & H_b^3(N, \partial N) \\
 \downarrow c & & \downarrow c & & \downarrow c \\
 H^3(\Gamma_\tau) & \xrightarrow[g]{\quad \cong \quad} & H^3(M_\tau) & \xrightarrow{F} & H^3(N, \partial N) \\
 & & \downarrow c & \nearrow (j_\tau, \varphi_\tau)^* & \\
 & & H^3(M_\tau, \mathbb{D}^2 \times S^1) & & \\
 & & \downarrow c & \nwarrow i^* & \\
 & & H^3(M_\tau, \mathbb{D}^2 \times S^1) & & 
 \end{array}$$

Since by naturality, all subdiagrams commute, the lemma follows.  $\square$

*Proof of Proposition 5.1.* Using the commutativity of the diagram (5.1) and the fact that  $c(\omega_3^b) = \omega_3$  and  $g \circ \rho^*(\omega_3) = \text{Vol}(\rho) \cdot \beta_{M_\tau}$  we compute

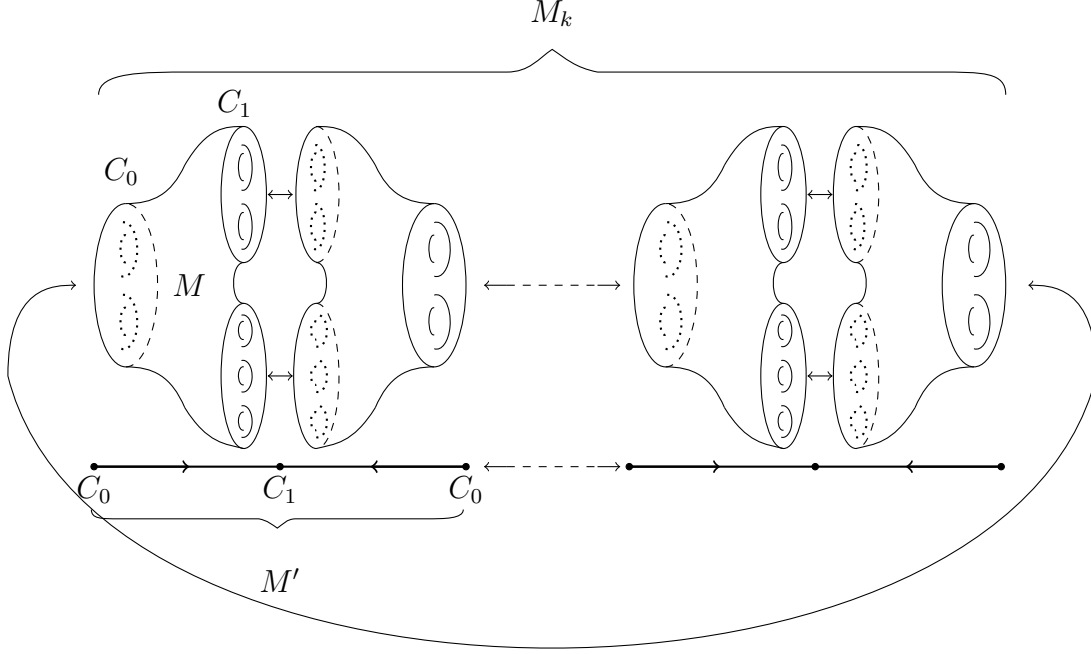
$$\begin{aligned}
 c \circ \Psi^{-1} \circ f_\tau^* \circ \rho^*(\omega_3^b) &= F \circ g \circ \rho^* \circ c(\omega_3^b) \\
 &= F \circ g \circ \rho^*(\omega_3) \\
 &= F(\text{Vol}(\rho) \cdot \beta_{M_\tau}) \\
 &= \text{Vol}(\rho) \cdot \beta_{[N, \partial N]}.
 \end{aligned}$$

It is immediate that

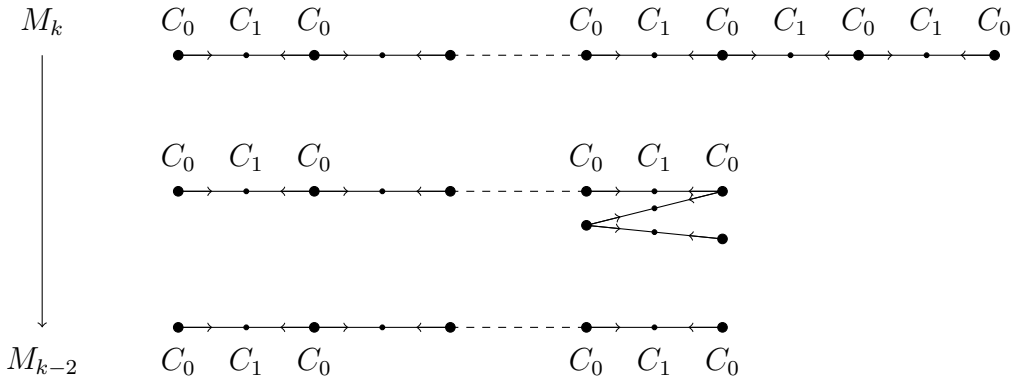
$$\text{Vol}(\rho_\tau) = \langle c \circ \Psi^{-1} \circ f^* \circ \rho^*(\omega_3^b), [N, \partial N] \rangle = \langle \text{Vol}(\rho) \cdot \beta_{[N, \partial N]}, [N, \partial N] \rangle = \text{Vol}(\rho),$$

which finishes the proof of the proposition.  $\square$

**5.2. Representations giving rational multiples of the maximal representation.** Let  $M$  be a  $n$ -dimensional hyperbolic manifold with nonempty totally geodesic boundary (possibly with cusps) (see for example [Mil76]). Suppose that the boundary of  $M$  has at least two connected components, which can be achieved by taking appropriate coverings of a manifold with a connected boundary. Decompose  $\partial M = C_0 \sqcup C_1$ . Let  $M'$  be the double of  $M$  along  $C_1$ . Observe that  $M'$  has as boundary two copies of  $C_0$  with opposite orientation. Glueing these two copies, we obtain a complete hyperbolic manifold  $M_1$ . We can repeat the procedure as follows: Take  $k$  copies of  $M'$ , glue the two copies of  $C_0$  two by two so as to obtain a connected closed hyperbolic manifold  $M_k$ . Observe that  $\text{vol}(M_k) = 2k \text{vol}(M)$ .



For any  $\ell < k$ , there are degree one maps  $f : M_k \rightarrow M_{k-\ell}$  obtained by folding  $\ell$  copies of  $M'$  in  $M_k$  along its boundary. The induced representation of  $\pi_1(M_k)$  obtained by the induced map on fundamental groups composed with the lattice embedding of  $\pi_1(M_{k-\ell})$  in  $\text{Isom}(\mathbb{H}^n)$  has volume equal to the volume of  $M_{k-\ell}$ , that is  $(k - \ell)/k$  times the volume of the maximal representation.



#### APPENDIX A. ON THE CONTINUITY

Let  $G$  be a locally compact group,  $\Gamma < G$  a lattice and  $L$  a locally compact group. We denote by  $C_b(X)$  the continuous bounded real valued functions on a

topological space  $X$ . We define a map

$$\begin{aligned} \mathrm{C}_b(L^{n+1}) \times \mathrm{Rep}(\Gamma, L) &\longrightarrow \mathrm{C}_b(\Gamma^{n+1}) \\ (c, \pi) &\longmapsto \pi^*(c), \end{aligned}$$

where

$$\pi^*(c)(\gamma_0, \dots, \gamma_n) = c(\pi(\gamma_0), \dots, \pi(\gamma_n)).$$

Observe that this map is continuous for the sup norm on  $\mathrm{C}_b(L^{n+1})$ , the pointwise convergence topology on  $\mathrm{Rep}(\Gamma, L)$  and the topology of pointwise convergence on  $\mathrm{C}_b(\Gamma^{n+1})$  with control of norms. Recall that in this topology a sequence  $\alpha_n \rightarrow \alpha$  if it converges pointwise and  $\sup_n \|\alpha_n\|_\infty < \infty$ .

We proceed to implement the transfer from  $\Gamma$  to  $G$ . For this, let  $s : \Gamma \backslash G \rightarrow G$  be a Borel section and  $r : G \rightarrow \Gamma$  be defined by

$$g = r(g) \cdot s(p(g)),$$

where  $p : G \rightarrow \Gamma \backslash G$  denotes the canonical projection. Given a  $\Gamma$ -invariant cochain  $\alpha \in \mathrm{C}_b(\Gamma^{n+1})^\Gamma$ , define

$$T\alpha(g_0, \dots, g_n) = \int_{\Gamma \backslash G} \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g),$$

where  $\mu$  is the Haar measure on  $G$  normalized so that  $\mu(\Gamma \backslash G) = 1$ .

**PROPOSITION A.1.** *Suppose that the Borel section has the property that images of compact subsets are precompact. Then*

- (1)  $T\alpha$  is continuous, hence  $T\alpha \in \mathrm{C}_b(G^{n+1})^G$ ,
- (2)  $T : \mathrm{C}_b(\Gamma^{n+1})^\Gamma \rightarrow \mathrm{C}_b(G^{n+1})^G$  is continuous for pointwise convergence on  $\Gamma$  with control of norms and uniform convergence on compact sets on  $G^{n+1}$ .

*Proof.* Let  $D = s(\Gamma \backslash G)$ . Choose  $\epsilon > 0$  and  $C \subset D$  compact with  $\mu(D \setminus C) < \epsilon$ . Then

$$|T\alpha(g_0, \dots, g_n) - \int_C \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g)| < \epsilon \|\alpha\|_\infty.$$

Now we write

$$\int_C \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g) = \sum_{\gamma_0, \dots, \gamma_n} \alpha(\gamma_0, \dots, \gamma_n) \mu(C \cap \gamma_0 D g_0^{-1} \cap \dots \cap \gamma_n D g_n^{-1}).$$

Before we continue with the proof of the proposition, we need to show that for every compact subset  $K \subset G$  the number  $F_K$  of translates of the fundamental domain  $D$  that  $K$  intersects is finite:

**LEMMA A.2.** *For any compact subset  $K \subset G$ , the set*

$$F_K := \{\gamma \in \Gamma \mid K \cap \gamma D \neq \emptyset\}$$

*is finite.*

Note that the lemma is wrong for arbitrary fundamental domains, even for cocompact  $\Gamma$ . Indeed, start by writing the standard fundamental domain  $(0, 1]$  of  $\mathbb{Z}$  in  $\mathbb{R}$  as

$$D_0 = \sqcup_{n=1}^{+\infty} (1/2^n, 1/2^{n-1}],$$

and perturb it by translating each of the disjoint interval of  $D_0$  by a different translation, for example obtaining the new fundamental domain

$$D = \sqcup_{n=1}^{+\infty} n + (1/2^n, 1/2^{n-1}].$$

Take as compact set the closed interval  $C = [0, 1]$ . Then for every  $-n \leq 0$ , the intersection  $C \cap (-n + D)$  is nonempty.

*Proof.* Set  $F := \cup_{\eta \in \Gamma} \eta K$  and observe that  $F \cap D = s(p(F))$  is relatively compact by our choice of Borel section. Since  $\gamma K \cap D = \gamma K \cap (F \cap D)$  and  $K$  and  $F \cap D$  are relatively compact, the lemma follows by the discreteness of  $\Gamma$ .  $\square$

Going back to the proof of the proposition, fix compact subsets  $C_0, \dots, C_n$  of  $G$  such that  $g_i \in C_i$ . Observe that  $F_{C_{g_i}} \subset F_{CC_i}$  and if  $\gamma_i \in F_{CC_i} \setminus F_{C_{g_i}}$  then the measure of  $C \cap \gamma_0 D g_0^{-1} \cap \dots \cap \gamma_n D g_n^{-1}$  is zero. We can thus rewrite the above sum as

$$\int_C \alpha(r(gg_0), \dots, r(gg_n)) d\mu(g) = \sum_{\gamma_i \in F_{CC_i}} \alpha(\gamma_0, \dots, \gamma_n) \mu(C \cap \gamma_0 D g_0^{-1} \cap \dots \cap \gamma_n D g_n^{-1}),$$

for any  $(g_0, \dots, g_n) \in C_0 \times \dots \times C_n$ .

The point (2) of Proposition follows since if  $\alpha_n \rightarrow \alpha$  with pointwise convergence and  $\sup_n \|\alpha_n\|_\infty < +\infty$  then  $T\alpha_n \rightarrow T\alpha$  uniformly on compact sets.

Finally, we show (1) by showing that the function

$$(g_0, \dots, g_n) \mapsto \mu(\{C \cap \gamma_0 D g_0^{-1} \cap \dots \cap \gamma_n D g_n^{-1}\})$$

is continuous. To estimate the difference

$$\mu(C \cap \bigcap_{i=1}^n \gamma_i D g_i^{-1}) - \mu(C \cap \bigcap_{i=1}^n \gamma_i D h_i^{-1})$$

we introduce the notation

$$A(x_0, \dots, x_n) := C \cap \bigcap_{i=0}^n \gamma_i D x_i^{-1},$$

for any  $x_0, \dots, x_n \in G$ . The above difference thus becomes

$$\mu(A(g_0, \dots, g_n)) - \mu(A(h_0, \dots, h_n))$$

which we rewrite as a telescopic sum

$$\sum_{i=0}^n \mu(A(h_0, \dots, h_{i-1}, g_i, g_{i+1}, \dots, g_n)) - \mu(A(h_0, \dots, h_{i-1}, h_i, g_{i+1}, \dots, g_n)).$$

Setting

$$B_j := C \cap \bigcap_{\ell=0}^{i-1} \gamma_\ell D h_\ell^{-1} \cap \bigcap_{\ell=i+1}^n \gamma_\ell D g_\ell^{-1},$$

the telescopic sum becomes

$$\sum_{i=0}^n \mu(B_j \cap \gamma_j D g_j^{-1}) - \mu(B_j \cap \gamma_j D h_j^{-1}).$$

Using the simple set theoretical inequality valid for any sets  $B, E, E'$

$$|\mu(B \cap E) - \mu(B \cap E')| \leq \mu((B \cap E) \Delta (B \cap E')) \leq \mu(E \Delta E'),$$

we obtain for each summand the estimate

$$|\mu(B_j \cap \gamma_j D g_j^{-1}) - \mu(B_j \cap \gamma_j D h_j^{-1})| \leq \mu(\gamma_j D g_j^{-1} \Delta \gamma_j D h_j^{-1}) = \|\chi_{D g_j^{-1} h_j} - \chi_D\|_1,$$

which finishes the proof of the proposition.  $\square$

*Proof of Theorem 1.2.* Consider  $\Gamma$  as a lattice in the full isometry group  $\text{Isom}(\mathbb{H}^n)$  and denote by  $\varepsilon : \text{Isom}(\mathbb{H}^n) \rightarrow \{-1, +1\}$  the homomorphism sending an isometry to  $+1$  if it preserves orientation and  $-1$  otherwise. By what precedes, the cohomology class  $\text{transf}(\rho^*(\omega_{\mathbb{H}^n}))$  can be represented by the continuous cocycle sending  $(g_0, \dots, g_n) \in \text{Isom}(\mathbb{H}^n)^{n+1}$  to

$$\int_{\Gamma \backslash \text{Isom}(\mathbb{H}^n)} \varepsilon(g) \omega_n(\rho(r(gg_0)), \dots, \rho(r(gg_n))) d\mu(g).$$

Note that the cocycle stays continuous after transferring from  $\text{Isom}^+(\mathbb{H}^n)$  to  $\text{Isom}(\mathbb{H}^n)$ . Integrating over a maximal compact subgroup  $K$  in  $\text{Isom}(\mathbb{H}^n)$  we obtain a continuous cocycle  $(\mathbb{H}^n)^{n+1} \rightarrow \mathbb{R}$  that sends an  $(n+1)$ -tuple of points  $g_0 K, \dots, g_n K \in \text{Isom}(\mathbb{H}^n)/K \cong \mathbb{H}^n$  to

$$(A.1) \quad \int_{K^{n+1}} \prod_{i=0}^n dk_i \int_{\Gamma \backslash \text{Isom}(\mathbb{H}^n)} \varepsilon(g) \omega_n(\rho(r(gg_0 k_0)), \dots, \rho(r(gg_n k_n))) d\mu(g).$$

We showed in [BBI13] that

$$\text{transf}(\rho^*(\omega_n)) = (\text{Vol}(\rho)/\text{vol}(M)) \cdot \omega_{\mathbb{H}^n} \in H_{\text{cb}}^n(\text{Isom}(\mathbb{H}^n), \mathbb{R}_\varepsilon).$$

Since there are no coboundaries in degree  $n$  for  $\text{Isom}(\mathbb{H}^n)$ -equivariant continuous bounded cochains on  $\mathbb{H}^n$ , this implies that we have a strict equality between (A.1) and

$$\frac{\text{Vol}(\rho)}{\text{vol}(M)} \cdot \omega_n(g_0 K, \dots, g_n K).$$

Since (A.1) varies continuously in  $\rho$ , so does  $\text{Vol}(\rho)$ .  $\square$

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